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Entropy of convex hulls—some Lorentz norm results

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Abstract

Let A be a subset of a type p Banach space E, $1<p\leqslant 2$, such that its entropy numbers satisfy $(\varepsilon_n(A))_n \in \ell_{q,s}$ for some $q,s\in(0,\infty)$. We show $(e_n(\operatorname{aco} A))_n \in \ell_{r,s}$ for the dyadic entropy numbers of the absolutely convex hull $\operatorname{aco} A$ of A, where r is defined by $1/r = 1/p' + 1/q$. Furthermore, we show for slowly decreasing entropy numbers that $(e_n(A))_n \in \ell_{q,s}$ implies $(e_n(\operatorname{aco} A))_n \in \ell_{p',s}$ for all $0<s<\infty$ and q defined by $1/q = 1/p' + 1/s$.
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1. Introduction and results

In the following, E always denotes a Banach space and B_E its closed unit ball. Given a bounded subset $A\subset E$, we define the entropy numbers of A to be

$$\varepsilon_n(A) := \inf\left\{\varepsilon>0 : \exists x_1,\dots,x_n\in A \text{ such that } A\subset \bigcup_{i=1}^{n}(x_i+\varepsilon B_E)\right\},\quad n\in\mathbb{N}.$$

Furthermore, the dyadic entropy numbers are $e_n(A) := \varepsilon_{2^{n-1}}(A)$, $n\geqslant 1$. It is common knowledge that if A is precompact, so is its absolutely convex hull $\operatorname{aco} A$. A problem

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which was first considered by Dudley [6] is to quantify this implication in terms of entropy numbers. In recent years this question has intensively been treated in different settings (cf. e.g. [1,3,5,7–9,13–15]). Furthermore, the "dual case" which leads to similar results has been considered in [2,4].

In order to state our results we have to recall some definitions: a Banach space E is said to be of type p, $1 \leqslant p \leqslant 2$, if there is a constant $C > 0$ such that for all $x_1, \ldots, x_n \in E$ we have the estimate

$$\int_0^1 \left\| \sum_{i=1}^n x_i r_i(t) \right\| dt \leqslant C \left(\sum_{i=1}^n \|x_i\|^p \right)^{1/p},$$

where (r_n) shall denote the Rademacher functions, i.e. $r_n(t) := \operatorname{sign}(\sin(2^n \pi t))$. The type p constant $\tau_p(E)$ is the smallest constant C satisfying the above inequality. If E is a Banach space of type p and $Z_1, \ldots, Z_n$ are independent E-valued random variables with finite pth moment the inequality

$$\mathbb{E} \left\| \sum_{i=1}^n (Z_i - \mathbb{E}Z_i) \right\| \leqslant 4\tau_p(E) \left(\sum_{i=1}^n \mathbb{E}\|Z_i\|^p \right)^{1/p} \tag{1}$$

holds (cf. [10]).

Furthermore, let $x = (x_i)$ be a sequence of real numbers. By $(s_n(x))$ we denote the non-increasing rearrangement of x, that is $s_n(x) := \inf\{c \geqslant 0 : \operatorname{card}\{i : |x_i| \geqslant c\} < n\}$. For $0 < p < \infty$ and $0 < q \leqslant \infty$ the Lorentz sequence space $\ell_{p,q}$ is then defined by

$$\ell_{p,q} := \{x : (n^{1/p-1/q} s_n(x)) \in \ell_q\},$$

which is equipped with the quasi-norm $\|x\|_{p,q} := \|(n^{1/p-1/q} s_n(x))\|_{\ell_q}$. For basic properties of these spaces we refer to [11]. As usual, we denote the conjugate of $p \in [1, \infty]$ by p', i.e. $1/p' := 1 - 1/p$. For two positive sequences (a_n) and (b_n) we write $a_n \preccurlyeq b_n$ if there exists a constant $c > 0$ such that $a_n \leqslant c b_n$ for all $n \in \mathbb{N}$. Moreover, we write $a_n \sim b_n$ if both $a_n \preccurlyeq b_n$ and $b_n \preccurlyeq a_n$.

Given a Banach space of type p, $1 < p \leqslant 2$, and a precompact subset A of E it was shown in [3] that

$$(\varepsilon_n(A))_n \in \ell_{q,\infty} \quad \Rightarrow \quad (e_n(\operatorname{aco} A))_n \in \ell_{r,\infty} \tag{2}$$

holds for all $q \in (0, \infty)$ if r is defined by $1/r = 1/p' + 1/q$. For Hilbert spaces, this implication was also shown in [1,15]. In [13,14] implication (2) was refined by establishing universal inequalities which imply

$$\varepsilon_n(A) \preccurlyeq n^{-1/q} \log(n+1)^\gamma \quad \Rightarrow \quad e_n(\operatorname{aco} A) \preccurlyeq n^{-1/q-1/p'} \log(n+1)^\gamma \tag{3}$$

for all $q \in (0, \infty)$ and all $\gamma \in \mathbb{R}$. Furthermore, it was shown in [14] that this is asymptotical optimal for certain sets $A \subset E$ whenever E has no type larger than p. Besides two inequalities (cf. [3]) for subsets A of Hilbert spaces and logarithmically decreasing $(\varepsilon_n(A))$ no sharp results on summability properties of $(e_n(\operatorname{aco} A))$ in terms of $(\varepsilon_n(A))$ are known so far. In particular, it is an open question whether (2) also holds for secondary indexes $s \neq \infty$. Before we positively answer this question we establish a universal inequality which estimates the entropy numbers of aco A in terms of the entropy numbers of A:

Theorem 1.1. *Let E be a Banach space of type $p\in(1,2]$ and $q\in(0,\infty)$. Then there exists a constant $c_q>0$ such that for all $n\geqslant 2$, all integers $\alpha_1<\ldots<\alpha_n$ and all bounded symmetric subsets $A\subset E$ we have*

$$e_{2m}(\operatorname{aco} A)\leqslant c_q m^{-1/q-1/p'} \sup_{i\leqslant \min\{m^{1+q/p'},\alpha_1\}} i^{1/q}\varepsilon_i(A) + 23\tau_p(E)2^{-n/p'}\left(\sum_{k=1}^{n}\left(2^{k/p'}\sum_{i=k}^{n}\varepsilon_{\alpha_i}(A)\right)^p\right)^{1/p},$$

where $m:=\lfloor 2^{n+2}\sum_{k=2}^{n}2^{-k}\log_2(\frac{2^{k+2}\alpha_k}{2^n}+3)\rfloor+2$.

Using Theorem 1.1 one can prove various known results on entropy numbers of convex hulls in type p spaces (cf. the examples below). Moreover, Theorem 1.1 leads to our main results:

Theorem 1.2. *Let E be a Banach space of type $p\in(1,2]$. For $q\in(0,\infty)$ define r by $1/r=1/p'+1/q$. Then for all bounded $A\subset E$ and all $s\in(0,\infty]$ we have*

$$(\varepsilon_n(A))_n\in\ell_{q,s} \;\Rightarrow\; (e_n(\operatorname{aco} A))_n\in\ell_{r,s}.$$

Since (3) is asymptotical optimal whenever E has no type larger than p it is obvious that Theorem 1.2 cannot be improved.

The following theorem provides a similar implication for subsets A with slowly decreasing entropy numbers.

Theorem 1.3. *Let E be a Banach space of type $p\in(1,2]$ and $s\in(0,\infty)$. Define q by $1/q=1/p'+1/s$. Then for all bounded subsets $A\subset E$ we have*

$$(e_n(A))_n\in\ell_{q,s} \;\Rightarrow\; (e_n(\operatorname{aco} A))_n\in\ell_{p',s}.$$

Again, the estimate of Theorem 1.3 cannot be improved if the type of E is exactly p (cf. Example 1.5).

As in [3] we say that a subset A of a Hilbert space H satisfies Dudley's entropy condition if $(e_n(A))_n\in\ell_{2,1}$. Recall, that the results of [3] (cf. Example 1.5) ensure that aco A satisfies Dudley's entropy condition provided that $(e_n(A))_n\in\ell_{r,\infty}$ for some r with $0<r<\frac{2}{3}$. By Theorem 1.3 this condition can be replaced by $(e_n(A))_n\in\ell_{2/3,1}$. As mentioned above the latter is optimal.

Finally, we provide some examples which demonstrate how recently proved results can be shown using Theorem 1.1. Recall, that all of these estimates are asymptotically optimal whenever E has no type larger than p (cf. [14]).

Example 1.4 (Carl et al. [3], Steinwart [13]). Let E be a Banach space of type p, $1 < p \leqslant 2$, $q > 0$ and $\gamma \in \mathbb{R}$. Fix an integer a with $a > q/p'$ and define $\alpha_k := 2^{n+ak}$. Then Theorem 1.1 yields implication (3) for all bounded subsets $A \subset E$.

Example 1.5 (Carl et al. [3], Steinwart [13]). Let E be a Banach space of type p, $1 < p \leqslant 2$, $A \subset E$ be a bounded subset, $q \in (0, p')$ and $\gamma \in \mathbb{R}$. Fix a with $q/p' < a < 1$ and define $\alpha_k := \lfloor 2^{n2^{(k-1)a}} \rfloor$. Then Theorem 1.1 yields

$$\begin{aligned} &\varepsilon_n(A) \preccurlyeq (\log(n+1))^{-1/q} (\log(\log(n+2)))^{\gamma} \\ &\quad \Rightarrow e_n(\operatorname{aco} A) \preccurlyeq n^{-1/p'} \log(n+1)^{1/p'-1/q} (\log(\log(n+2)))^{\gamma}. \end{aligned}$$

Example 1.6 (Gao [7], Creutzig and Steinwart [5]). Let E be a Banach space of type p, $1 < p \leqslant 2$, and $A \subset E$ be a bounded subset. We define $\alpha_k := 2^{n2^{k-1}}$ or $\alpha_k := 2^{2^k}$. Then in both cases Theorem 1.1 yields

$$\varepsilon_n(A) \preccurlyeq (\log(n+1))^{-1/p'} \quad \Rightarrow \quad e_n(\operatorname{aco} A) \preccurlyeq n^{-1/p'} \log(n+1).$$

Example 1.7 (Carl et al. [3], Steinwart [13]). Let E be a Banach space of type p, $1 < p \leqslant 2$, $A \subset E$ be a bounded subset and q with $p' < q < \infty$. We fix an α with $1 < \alpha < q/p'$ and define $\alpha_k := \lfloor 2^{2^{\alpha k}} \rfloor$. Then the theorem yields

$$\varepsilon_n(A) \preccurlyeq (\log(n+1))^{-1/q} \quad \Rightarrow \quad e_n(\operatorname{aco} A) \preccurlyeq n^{-1/q}.$$

2. Proof of the results

We need the following result which was shown in [13,14].

Theorem 2.1. *Let E be a Banach space of type p, $1 < p \leqslant 2$, and $q \in (0, \infty)$. Then there exists a constant $c_q > 0$ such that for all bounded symmetric subsets $A \subset E$ with $\operatorname{card}(A) \leqslant n$ and all $k \geqslant 1$ we have*

$$e_k(\operatorname{aco} A) \leqslant c_q k^{-1/q-1/p'} \sup_{i \leqslant \min\{k^{1+q/p'}, n\}} i^{1/q} \varepsilon_i(A).$$

In order to prove Theorem 1.1 we also need a refinement of the decomposition techniques of [3,13] which is stated in the following lemma:

Lemma 2.2. *Let $A \subset E$, $n \in \mathbb{N}$ and $\alpha_1, \ldots, \alpha_n$ be integers. Then there exist a sequence $\mathscr{P}_1, \ldots, \mathscr{P}_n$ of partitions of A and elements $x_i^{(k)} \in A$, $i = 1, \ldots, |\mathscr{P}_k|$, $k = 1, \ldots, n$, such* *that $\mathscr{P}_k$ is finer than $\mathscr{P}_{k-1}$ for all $k = 2, \ldots n$ and*

$$|\mathscr{P}_k| \leqslant \alpha_k \quad \textit{for all } k = 1, \ldots, n,$$

$$P_i^k \subset x_i^{(k)} + \left(2 \sum_{l=k}^{n} \varepsilon_{\alpha_l}(A)\right) B_E \quad \textit{for all } P_i^k \in \mathscr{P}_k.$$

Proof. The construction is based on a kind of backwards induction. By the definition of $\varepsilon_{\alpha_n}(A)$ there is a minimal $2\varepsilon_{\alpha_n}(A)$-net $B = \{x_1^{(n)}, \ldots, x_m^{(n)}\} \subset A$ of A with $m := |B| \leqslant \alpha_n$. Let $\mathscr{P}_n = \{P_1^n, \ldots, P_m^n\}$ be a partition with $P_i^n \subset x_i^{(n)} + 2\varepsilon_{\alpha_n}(A) B_E$ for all $i = 1, \ldots, m$.

Now, let us suppose that we already have constructed $\mathscr{P}_k$ and the corresponding elements $x_i^{(k)}$. In particular, we have

$$P_i^k \subset x_i^{(k)} + \left(2 \sum_{l=k}^{n} \varepsilon_{\alpha_l}(A)\right) B_E.$$

By the definition of $\varepsilon_{\alpha_{k-1}}(A)$ there is a minimal $2\varepsilon_{\alpha_{k-1}}(A)$-net $B = \{x_1^{(k-1)}, \ldots, x_m^{(k-1)}\} \subset A$ of A with cardinality $m := |B| \leqslant \alpha_{k-1}$. Let $A_1, \ldots, A_m$ be a partition of A with $A_i \subset x_i^{(k-1)} + 2\varepsilon_{\alpha_{k-1}}(A) B_E$. For $1 \leqslant i \leqslant m$ we then define

$$P_i^{k-1} := \bigcup_{\substack{j \text{ with} \\ x_j^{(k)} \in A_i}} P_j^k$$

if there is an index j with $x_j^{(k)} \in A_i$. We denote the collection of these P_i^{k-1}'s by $\mathscr{P}_{k-1}$. Clearly, $\mathscr{P}_{k-1}$ is a partition of A with $|\mathscr{P}_{k-1}| \leqslant m \leqslant \alpha_{k-1}$ and $\mathscr{P}_k$ is finer than $\mathscr{P}_{k-1}$. To check the last assertion we chose an arbitrary $x \in P_i^{k-1}$. Then there is an index j with $x_j^{(k)} \in A_i$ and $x \in P_j^k$. Since $||x_j^{(k)} - x_i^{(k-1)}|| \leqslant 2\varepsilon_{\alpha_{k-1}}(A)$ we obtain

$$\begin{aligned} ||x - x_i^{(k-1)}|| &\leqslant ||x - x_j^{(k)}|| + ||x_j^{(k)} - x_i^{(k-1)}|| \\ &\leqslant 2 \sum_{l=k}^{n} \varepsilon_{\alpha_l}(A) + 2\varepsilon_{\alpha_{k-1}}(A) = 2 \sum_{l=k-1}^{n} \varepsilon_{\alpha_l}(A). \qquad \square \end{aligned}$$

Proof of Theorem 1.1. Let $\mathscr{P}_1, \ldots, \mathscr{P}_n$ be a sequence of partitions according to Lemma 2.2. Using backwards induction, we find elements $y_j^{(k)} \in P_j^k$, $k = 1, \ldots, n$ and $j = 1, \ldots, |\mathscr{P}_k|$, with

$$y_j^{(n)} = x_j^{(n)} \quad \text{for } j = 1, \ldots, |\mathscr{P}_n|$$

and

$$y_j^{(k)} = y_i^{(k+1)} \quad \text{for } k = 1, \ldots, n-1,\ j = 1, \ldots, |\mathscr{P}_k|$$

and an index i with $P_i^{k+1} \subset P_j^k$.

Now, for $1 \leqslant k \leqslant n$ and $x \in A$ we define

$$t_k(x) := y_j^{(k)}$$

if $x \in P_j^k$. Then, our construction guarantees that the cardinality of

$$D_k := \{t_k(x) - t_{k-1}(x) : x \in A\}, \quad k = 2, \dots, n$$

can be estimated by $|D_k| \leqslant |\mathscr{P}_k| \leqslant \alpha_k$. Of course, for $D_1 := \{t_1(x) : x \in A\}$ this is also true. Moreover, for all $x \in A$ and every $k = 2, \dots, n$ we have $t_k(x), t_{k-1}(x) \in P_j^{k-1}$ for a suitable index j. Hence, we find

$$||D_k|| \leqslant 4 \sum_{i=k-1}^{n} \varepsilon_{\alpha_i}(A) \tag{4}$$

by the definition of $\mathscr{P}_{k-1}$. Here, we write $||D_k|| := \sup_{x \in D_k} ||x||$ for short. After symmetrizing $D'_k := D_k \cup (-D_k) \cup \{0\}$ we define $C_k := \operatorname{aco} D_k = \operatorname{co} D'_k$ and $E_n := \sum_{k=2}^{n} C_k$. Our construction guarantees $\operatorname{aco}\{x_1^{(n)}, \dots, x_{|\mathscr{P}_n|}^{(n)}\} \subset C_1 + E_n$ and hence $C_1 + E_n$ is a $2\varepsilon_{\alpha_n}(A)$-net of aco A. In particular, we have

$$e_{2m}(\operatorname{aco} A) \leqslant e_m(C_1) + e_m(E_n) + 2\varepsilon_{\alpha_n}(A). \tag{5}$$

Obviously, we obtain

$$2\varepsilon_{\alpha_n}(A) \leqslant 23\tau_p(E) 2^{-n/p'}((2^{n/p'}\varepsilon_{\alpha_n}(A))^p)^{1/p}. \tag{6}$$

Moreover, $D_1 \subset A$, $|D_1| \leqslant \alpha_1$ and Theorem 2.1 imply

$$e_m(C_1) \leqslant c_q m^{-1/q-1/p'} \sup_{i \leqslant \min\{m^{1+q/p'}, \alpha_1\}} i^{1/q} \varepsilon_i(A). \tag{7}$$

Now, we estimate $e_m(E_n)$ using an argument of [7] (cf. also [5]) which originally goes back to Maurey (cf. [12]): we write $D'_k = \{x_1^{(k)}, \dots, x_{d_k}^{(k)}\} \cup \{0\}$ for $k = 2, \dots, n$. Then every $z_k \in C_k$ can be represented by

$$z_k = \sum_{i=1}^{d_k} a_i^{(k)} x_i^{(k)} \quad \text{where } a_i^{(k)} \geqslant 0 \text{ and } \sum_{i=1}^{d_k} a_i^{(k)} \leqslant 1.$$

Let Z_k be a random vector with range D'_k and

$$\mathbb{P}(Z_k = x_i^{(k)}) = a_i^{(k)}, \quad i = 1, \dots, x_{d_k}^{(k)},$$

$$\mathbb{P}(Z_k = 0) = 1 - \sum_{i=1}^{d_k} a_i^{(k)}.$$

It is trivial to obtain $\mathbb{E} Z_k = z_k$. For brevity's sake we write $m_k := 2^{n-k}$ for $k = 2, \dots, n$. Now, we take independent random vectors $Z_{2,1}, \dots, Z_{2,m_1}, \dots, Z_{n,1}, \dots, Z_{n,m_n}$ such that $Z_{k,i}$ is a copy of Z_k for all $k = 2, \dots, n$ and $i = 1, \dots, m_k$. With $Y_{k,i} := \frac{1}{m_k} Z_{k,i}$

and inequality (1) we then obtain

$$\begin{aligned}
\mathbb{E}\left\|\sum_{k=2}^{n} z_k - \sum_{k=2}^{n} \frac{1}{m_k} \sum_{i=1}^{m_k} Z_{k,i}\right\| &= \mathbb{E}\left\|\sum_{k=2}^{n} \sum_{i=1}^{m_k} (\mathbb{E}\, Y_{k,i} - Y_{k,i})\right\| \\
&\leqslant 4\, \tau_p(E) \left(\sum_{k=2}^{n} \sum_{i=1}^{m_k} \mathbb{E}\|\, Y_{k,i}\|^p\right)^{1/p} \\
&\leqslant 4\, \tau_p(E) \left(\sum_{k=2}^{n} (m_k^{-1/p'} \|D_k{}'\|)^p\right)^{1/p} \\
&\leqslant 16\, \tau_p(E) 2^{-n/p'} \left(\sum_{k=2}^{n} \left(2^{k/p'} \sum_{i=k-1}^{n} \varepsilon_{\alpha_i}(A)\right)^p\right)^{1/p} \\
&\leqslant 23\, \tau_p(E) 2^{-n/p'} \left(\sum_{k=1}^{n-1} \left(2^{k/p'} \sum_{i=k}^{n} \varepsilon_{\alpha_i}(A)\right)^p\right)^{1/p} \\
&=: \varepsilon_0.
\end{aligned}$$

Because the expectation is less than ε_0, there is a realization of the $Z_{k,i}$ for which the inequality also holds. In particular, the set

$$X := \left\{\sum_{k=2}^{n} \frac{1}{m_k} \sum_{i=1}^{m_k} d_{k,i} : d_{k,i} \in D'_k\right\}$$

of all possible realizations of random sums $\sum_{k=1}^{n} \frac{1}{m_k} \sum_{i=1}^{m_k} Z_{k,i}$, where $Z_{k,i} \in D'_k$ are arbitrary random vectors, form a ε_0-net of E_n. With Stirling's formula, we find

$$\begin{aligned}
\log_2 |X| &\leqslant \sum_{k=2}^{n} \log_2 \binom{|D'_k| + m_k - 1}{m_k} \\
&\leqslant 3 \sum_{k=2}^{n} m_k \log_2 \left(\frac{|D'_k|}{m_k} + 3\right) \\
&\leqslant 2^{n+2} \sum_{k=2}^{n} 2^{-k} \log_2 \left(\frac{2^{k+2} \alpha_k}{2^n} + 3\right).
\end{aligned}$$

Therefore, we obtain

$$e_m(E_n) \leqslant \varepsilon_{|X|}(E_n) \leqslant 23\, \tau_p(E) 2^{-n/p'} \left(\sum_{k=1}^{n-1} \left(2^{k/p'} \sum_{i=k}^{n} \varepsilon_{\alpha_i}(A)\right)^p\right)^{1/p},$$

which together with (5), (6) and (7) completes the proof. □

Proof of Theorem 1.2. Obviously, we may assume without loss of generality that A is symmetric. Furthermore, we only have to show the assertion for $s < \infty$. Let us fix an integer a with $a > 2q/p'$. We define $\alpha_k := 2^{n+ak}$ for $k = 1, \ldots, n$.

Then we obtain

$$\begin{aligned} m &= \left\lfloor 2^{n+2} \sum_{k=2}^{n} 2^{-k} \log_2\left(\frac{2^{k+2}\alpha_k}{2^n} + 3\right) \right\rfloor + 2 \\ &= \left\lfloor 2^{n+2} \sum_{k=2}^{n} 2^{-k} \log_2(2^{(a+1)k+2} + 3) \right\rfloor + 2 \\ &\leqslant 2^{n+2} \sum_{k=2}^{n} 2^{-k}((a+1)k+3) + 2 \\ &\leqslant c_1 2^n \end{aligned}$$

for a suitable constant $c_1 > 0$ independent of n. Analogously, we find a constant $c_1' > 0$ such that $m \geqslant c_1' 2^n$. Furthermore, we have

$$\begin{aligned} \left(\sum_{k=1}^{n} \left(2^{k/p'} \sum_{i=k}^{n} \varepsilon_{\alpha_i}(A)\right)^p\right)^{1/p} &\leqslant \sum_{k=1}^{n} 2^{k/p'} \sum_{i=k}^{n} \varepsilon_{\alpha_i}(A) \\ &= \sum_{i=1}^{n} \varepsilon_{\alpha_i}(A) \sum_{k=1}^{i} 2^{k/p'} \\ &\leqslant c_2 \sum_{i=1}^{n} 2^{i/p'} e_{n+1+ai}(A) \end{aligned}$$

for a constant $c_2 > 0$ only depending on p'. Therefore, Theorem 1.1 yields

$$e_{2c_1 2^n}(\operatorname{aco} A) \leqslant c_3 2^{-(1/p'+1/t)n} \sup_{i \leqslant 2^n} i^{1/t} \varepsilon_i(A) + c_4 2^{-n/p'} \sum_{i=1}^{n} 2^{i/p'} e_{n+1+ai}(A) \tag{8}$$

for all $n \geqslant 2$, all $t > 0$ and constants $c_3, c_4 > 0$ independent of n and A. Since the assertion is equivalent to $(2^{n/p'+n/q} e_{2c_1 2^n}(\operatorname{aco} A))_n \in \ell_s$ it suffices to show

$$\left(2^{n/q-n/t} \sup_{k \leqslant 2^n} k^{1/t} \varepsilon_k(A)\right)_n \in \ell_s, \tag{9}$$

$$\left(2^{n/q} \sum_{i=1}^{n} 2^{i/p'} e_{n+1+ai}(A)\right)_n \in \ell_s \tag{10}$$

for a suitable $t > 0$. Let us fix a $t > 0$ with $t < q$. Then $(\varepsilon_k(A))_k \in \ell_{q,s}$ implies

$$\left(\left(\frac{1}{n} \sum_{k=1}^{n} \varepsilon_k^t(A)\right)^{1/t}\right)_n \in \ell_{q,s}. \tag{11}$$

Since $n^{-1/t} \sup_{k \leqslant n} k^{1/t} \varepsilon_k(A) \leqslant (\frac{1}{n} \sum_{k=1}^{n} \varepsilon_k^t(A))^{1/t}$ for all $n \geqslant 1$ relation (11) implies $(n^{-1/t} \sup_{k \leqslant n} k^{1/t} \varepsilon_k(A))_n \in \ell_{q,s}$. Since, the latter sequence is decreasing this is equivalent to (9). Now, let us treat (10): If $s > 1$ we define $b := 2s/p'$ and $t :=$

$1/(s-1)$. Then, we observe

$$\begin{aligned}\left(\sum_{i=1}^{n} 2^{i/p'} e_{n+1+ai}(A)\right)^{s} &= \left(\sum_{i=1}^{n} (2^{is/p'-ib} 2^{ib} e^{s}_{n+1+ai}(A))^{1/s}\right)^{s} \\ &\leqslant \left(\sum_{i=1}^{n} (2^{is/p'-ib})^{t}\right)^{1/t} \sum_{i=1}^{n} 2^{ib} e^{s}_{n+1+ai}(A) \\ &\leqslant c_5 \sum_{i=1}^{n} 2^{ib} e^{s}_{n+1+ai}(A)\end{aligned}$$

for a constant $c_5 \geqslant 1$ only depending on s and p'. If $0 < s \leqslant 1$ we also define $b := 2s/p'$ and find

$$\left(\sum_{i=1}^{n} 2^{i/p'} e_{n+1+ai}(A)\right)^{s} \leqslant \sum_{i=1}^{n} 2^{is/p'} e^{s}_{n+1+ai}(A) \leqslant \sum_{i=1}^{n} 2^{ib} e^{s}_{n+1+ai}(A).$$

Hence for all $s > 0$ and $b := 2s/p'$ we obtain

$$\begin{aligned}\sum_{n=1}^{N}\left(2^{n/q} \sum_{i=1}^{n} 2^{i/p'} e_{n+1+ai}(A)\right)^{s} &\leqslant c_5 \sum_{n=1}^{N} 2^{ns/q} \sum_{i=1}^{n} 2^{ib} e^{s}_{n+1+ai}(A) \\ &= c_5 \sum_{i=1}^{N} 2^{ib} \sum_{n=i}^{N} 2^{ns/q} e^{s}_{n+1+ai}(A) \\ &= c_5 \sum_{i=1}^{N} 2^{ib} \sum_{n=(a+1)i}^{N+ai} 2^{(n-ia)s/q} e^{s}_{n+1}(A) \\ &\leqslant c_5 \sum_{i=1}^{N} 2^{i(b-as/q)} \sum_{n=1}^{(a+1)N} 2^{ns/q} e^{s}_{n+1}(A) \\ &\leqslant c_6 \sum_{n=1}^{(a+1)N} 2^{ns/q} e^{s}_{n}(A)\end{aligned}$$

for a constant $c_6 > 0$ independent of N and A. □

Proof of Theorem 1.3. Let us fix an $\varepsilon > 0$ with $\varepsilon s < p'$. This definition implies $(1+\varepsilon)s < p' + s$ and hence we can chose an a with $(1+\varepsilon)s/(s+p') < a < 1$. We define $\alpha_k := \lfloor 2^{n2^{a(k-1)}} \rfloor$ for $k = 1, \ldots, n$. Then it is easily checked that there exists constants $c_1, c_2 > 0$ independent of n with $c_2 n 2^n \leqslant m \leqslant c_1 n 2^n$. As in the proof of Theorem 1.2 we also find

$$\begin{aligned}e_{2c_1 n 2^n}(\operatorname{aco} A) \leqslant{}& c_3 2^{-(1/p'+1/r)n} n^{-(1/p'+1/r)} \sup_{i \leqslant 2^n} i^{1/r} \varepsilon_i(A) \\ &+ c_4 2^{-n/p'} \sum_{i=1}^{n} 2^{i/p'} e_{\lfloor n 2^{a(i-1)} \rfloor + 1}(A)\end{aligned}$$

for all $n \geqslant 2$, all $r > 0$ and constants $c_3, c_4 > 0$ independent of n and A. In particular, there exist constants $c_5, c_6 > 0$ independent of n and A with

$$e_{c_1 2^n}(\text{aco}\, A) \leqslant c_5 2^{-(1/p'+1/r)n} \sup_{k \leqslant 2^n} k^{1/r} \varepsilon_k(A) + c_6 2^{-n/p'} n^{1/p'} \sum_{i=1}^{n} 2^{i/p'} e_{\lfloor n 2^{a(i-1)-1} \rfloor + 1}(A) \tag{12}$$

for all $n \geqslant 2$ and all $r > 0$. Since the assertion is equivalent to $(2^{n/p'} e_{2^n}(\text{aco}\, A))_n \in \ell_s$ it suffices to show

$$\left(2^{-n/r} \sup_{k \leqslant 2^n} k^{1/r} \varepsilon_k(A) \right)_n \in \ell_s \tag{13}$$

$$\left(n^{1/p'} \sum_{i=1}^{n} 2^{i/p'} e_{\lfloor n 2^{a(i-1)-1} \rfloor + 1}(A) \right)_n \in \ell_s \tag{14}$$

for a suitable $r > 0$. In order to show (13) recall that $(e_k(A))_k \in \ell_{q,s}$ implies $\varepsilon_k(A) \preccurlyeq (\log(n+1))^{-1/q}$. Hence, we find

$$2^{-n/r} \sup_{k \leqslant 2^n} k^{1/r} \varepsilon_k(A) \preccurlyeq 2^{-n/r} \sup_{k \leqslant 2^n} k^{1/r} (\log(n+1))^{-1/q} \preccurlyeq n^{-1/q}.$$

Since $s > q$ this implies (13).

Now, let us treat (14): if $s > 1$ define $b := (1+\varepsilon)s/p'$ and $t := 1/(s-1)$. Then, we observe

$$\begin{aligned}
\left(\sum_{i=1}^{n} 2^{i/p'} e_{\lfloor n 2^{a(i-1)-1} \rfloor + 1}(A) \right)^s &= \left(\sum_{i=1}^{n} (2^{is/p'-ib} 2^{ib} e^s_{\lfloor n 2^{a(i-1)-1} \rfloor + 1}(A))^{1/s} \right)^s \\
&\leqslant \left(\sum_{i=1}^{n} (2^{is/p'-ib})^t \right)^{1/t} \sum_{i=1}^{n} 2^{ib} e^s_{\lfloor n 2^{a(i-1)-1} \rfloor + 1}(A) \\
&\leqslant c_7 \sum_{i=1}^{n} 2^{ib} e^s_{\lfloor n 2^{a(i-1)-1} \rfloor + 1}(A)
\end{aligned}$$

for a constant $c_7 \geqslant 1$ independent of n and A. If $0 < s \leqslant 1$ we also define $b := (1+\varepsilon)s/p'$. Then, we find

$$\begin{aligned}
\left(\sum_{i=1}^{n} 2^{i/p'} e_{\lfloor n 2^{a(i-1)-1} \rfloor + 1}(A) \right)^s &\leqslant \sum_{i=1}^{n} 2^{is/p'} e^s_{\lfloor n 2^{a(i-1)-1} \rfloor + 1}(A) \\
&\leqslant \sum_{i=1}^{n} 2^{ib} e^s_{\lfloor n 2^{a(i-1)-1} \rfloor + 1}(A).
\end{aligned}$$

Hence for all $s>0$ and $b := (1+\varepsilon)s/p'$ we obtain

$$\begin{aligned}\sum_{n=1}^{N}\left(n^{1/p'}\sum_{i=1}^{n}2^{i/p'}e_{\lfloor n2^{a(i-1)-1}\rfloor+1}(A)\right)^{s} &\leqslant c_7\sum_{n=1}^{N}n^{s/p'}\sum_{i=1}^{n}2^{ib}e^{s}_{\lfloor n2^{a(i-1)-1}\rfloor+1}(A)\\ &= c_7\sum_{i=1}^{N}2^{ib}\sum_{n=i}^{N}n^{s/p'}e^{s}_{\lfloor n2^{a(i-1)-1}\rfloor+1}(A)\\ &\leqslant c_8\sum_{i=1}^{N}2^{ib-ia-ias/p'}\sum_{n=1}^{\infty}n^{s/p'}e^{s}_{n}(A)\\ &\leqslant c_9\sum_{n=1}^{\infty}(n^{1/q-1/s}e_n(A))^{s}\end{aligned}$$

for constants $c_8, c_9>0$ independent of N and A. □

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